General Relativity $\boxed{\text{Week 13}}$

Last time, we proved the local well-posedness theorem for quasilinear wave equations:

Thm: For any fixed $k > 2n+2$, the initial value problem

$$\begin{cases} G^{\alpha\beta}(\psi)\,\partial_\alpha\partial_\beta\psi = N(\psi,\partial\psi) & \text{on } \mathbb{R}^{n+1} \\ \psi|_{t=0} = \psi_0, \quad \partial_t\psi|_{t=0} = \psi_1 \end{cases}$$

with $|G^{\alpha\beta} - m^{\alpha\beta}| < \frac{1}{10 \cdot n}$ satisfies the following:

If $M := \|\psi_0\|^2_{H^k} + \|\psi_1\|^2_{H^{k-1}}$, there exists $T = T(M) > 0$ and a <u>unique</u> solution ψ on $[0,T] \times \mathbb{R}^n$ satisfying

$$\cancel{\sup}\ \|\psi\|^2_{L^\infty_t H^k_x} + \|\partial_t\psi\|^2_{L^\infty_t H^{k-1}_x} \leq C(M).$$

Moreover, this solution depends continuously on the initial data, in the sense that if for a sequence:

$$\|\psi_0^{(n)} - \psi_0\|_{H^k} + \|\psi_1^{(n)} - \psi_1\|_{H^{k-1}} \xrightarrow{n\to\infty} 0,$$

then $T \leq \liminf_n T^{(n)}$ and

$$\|\psi^{(n)} - \psi\|_{L^\infty_t H^k} + \|\partial_t\psi^{(n)} - \partial_t\psi\|_{L^\infty_t H^{k-1}} \xrightarrow{n} 0.$$

(The last statement: Cauchy stability).

By gluing together local solutions:

Corollary: Let $\Sigma^n \subset M^{n+1}$ and consider the initial value problem:

$$\begin{cases} \Box_{g(x,\psi)}\psi = N(x,\psi,d\psi) \\ \psi|_\Sigma = \psi_0, \quad n(\psi)|_\Sigma = \psi_1, \end{cases}$$

$N: M \times \mathbb{R} \times T^*M \to \mathbb{R}$ smooth,
$g(x,\cdot): \mathbb{R} \to T^*_xM \otimes T^*_xM$
(where n a fixed vector field transversal to Σ)

So that $g(x,\psi)$: Lorentzian and the initial data are such that $g(x,\psi_0)|_\Sigma$ is Riemannian.

Then $\exists\, U \subseteq M$ open containing Σ and a unique solution ψ on U (assumption: $(\psi_0, \psi_1) \in C^\infty(\Sigma) \times C^\infty(\Sigma)$).

The vacuum Einstein equations:

When $n \geq 2$: $Ric - \frac{1}{2} R \cdot g = 0 \iff Ric = 0$. We would like to construct solutions. From the expression $R^a{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^a_{\delta\beta} - \partial_\delta \Gamma^a_{\gamma\beta} + \Gamma \cdot \Gamma$, we compute:

$$Ric_{\mu\nu} = -\frac{1}{2} g^{a\beta} \left(\partial_a \partial_\beta g_{\mu\nu} + \partial_\mu \partial_\nu g_{a\beta} - \partial_a \partial_\mu g_{\beta\nu} - \partial_\beta \partial_\nu g_{a\mu} \right) + N_{\mu\nu}(g, \partial g).$$

If it was just the first term (namely $g^{a\beta} \partial_a \partial_\beta g_{\mu\nu}$) then $Ric = 0$ would be of the form $\Box_g g_{\mu\nu} = \tilde{N}_{\mu\nu}(g, \partial g)$, so we could construct solutions by solving a regular initial value problem. But this is not true in a general coordinate system!

(if it was true in any coordinate system, we would have uniqueness of solutions $g_{a\beta}$ - but this cannot be true, since, by doing a non-trivial coordinate transformation, I get a new expression $g'_{a\beta}$ of the same metric, which still solves the equations).

Can we fix a "good" coordinate system, in which only the first term in the above expansion survives?

Wave coordinates: $\Box_g x^\mu = 0 \iff g^{a\beta} \Gamma^\mu_{a\beta} = 0$

Setting $\Gamma_\mu = g_{\mu\nu} g^{a\beta} \Gamma^\nu_{a\beta} = g^{a\beta} \left(\partial_a g_{\mu\beta} - \frac{1}{2} \partial_\mu g_{a\beta} \right)$

(Note: This is not a 1-form, it is coordinate dependent)

Then $\Box_g x^\mu = 0$ for all $\mu = 0, 1, \dots, n \iff \reflectbox{∂}_\mu = 0 \quad \forall \mu = 0, 1, \dots, n.$

And:
$$Ric_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} + \frac{1}{2}(\partial_\mu \reflectbox{∂}_\nu + \partial_\nu \reflectbox{∂}_\mu) + N(g, \partial g)$$

<u>Def:</u> Reduced Ricci "tensor":
$$\widetilde{Ric}_{\mu\nu} = Ric_{\mu\nu} - \frac{1}{2}(\partial_\mu \reflectbox{∂}_\nu + \partial_\nu \reflectbox{∂}_\mu)$$

So: If $\widetilde{Ric}_{\mu\nu} = 0$ <u>and</u> $\reflectbox{$\partial$}_\mu = 0 \implies Ric_{\mu\nu} = 0.$

Do we have a good equation for $\reflectbox{$\partial$}_\mu$? In general no, but:

<u>Lemma:</u> If $\widetilde{Ric}_{\mu\nu} = 0$ then $\Box_g \reflectbox{$\partial$}_\mu + A_\mu^{\alpha\beta} \partial_\alpha \reflectbox{∂}_\beta + B_\mu^\gamma \reflectbox{∂}_\gamma = 0$

<u>Proof:</u> $\widetilde{Ric} = 0 \iff Ric_{\mu\nu} = \frac{1}{2}(\partial_\mu \reflectbox{∂}_\nu + \partial_\nu \reflectbox{∂}_\mu) = \frac{1}{2}(\nabla_\mu \reflectbox{∂}_\nu + \nabla_\nu \reflectbox{∂}_\mu) + \Gamma \cdot \reflectbox{∂}$

and $R = g^{\mu\nu} Ric_{\mu\nu} = \nabla^\alpha \reflectbox{∂}_\alpha + g \cdot \Gamma \cdot \reflectbox{∂}$

2^{nd} Bianchi identity: Implies that
$$0 = \nabla^\mu (Ric_{\mu\nu} - \frac{1}{2} R \cdot g_{\mu\nu}) = \nabla^\mu Ric_{\mu\nu} - \frac{1}{2} \nabla_\nu R$$

$$\implies 0 = \frac{1}{2} \nabla^\mu (\nabla_\mu \reflectbox{∂}_\nu + \nabla_\nu \reflectbox{∂}_\mu) + l.o.t - \frac{1}{2} \nabla_\nu (\nabla^\alpha \reflectbox{∂}_\alpha) + l.o.t$$

$$= \frac{1}{2} \Box_g \reflectbox{∂}_\nu + \underbrace{\frac{1}{2} \nabla^\mu \nabla_\nu \reflectbox{∂}_\mu - \frac{1}{2} \nabla_\nu \nabla^\mu \reflectbox{∂}_\mu}_{= g^{\alpha\beta} R_{\nu\alpha}{}^\gamma{}_\beta \reflectbox{∂}_\gamma \, = l.o.t.} + l.o.t = 0$$

$\Box$

So: A strategy to construct solutions of $Ric = 0$:

- Solve $\widetilde{Ric} = 0$ (which is of the form $\Box_g g_{\mu\nu} = N_{\mu\nu}(g, \partial g)$)

with initial data
$$\left.\begin{array}{l} g_{\mu\nu}|_{t=0}, \ \partial_t g_{\mu\nu}|_{t=0}, \\ \text{and } \ \reflectbox{∂}_\mu|_{t=0} = 0, \ \partial_t \reflectbox{∂}_\mu|_{t=0} \end{array}\right\} \text{overdetermined!!}$$

(Note: Since λ solves a wave equation, these initial data for λ imply that $\lambda \equiv 0$)

Is there a way to untangle the overdetermined nature of the problem?

Let us introduce a more geometric viewpoint:

Let (M, g) be a Lorentzian manifold satisfying $Ric_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}$.

Let $\Sigma \subset M$ be a spacelike hypersurface.

The <u>induced metric</u> $\bar{g}$ and <u>second fundamental form</u> k (with respect to a unit normal $\hat{n}$: $k(X,Y) = g(\nabla_X \hat{n}, Y)$) satisfy the Gauss-Codazzi equations, from which we obtain the <u>constraint equations</u>:

$$
① \begin{cases} \bar{R} + \underbrace{(tr_{\bar{g}} k)^2 - \|k\|_{\bar{g}}^2}_{-\|k \wedge k\|_{\bar{g}}^2} = 16\pi T(\hat{n}, \hat{n}) & \longrightarrow \text{Hamiltonian constraint} \\ \underbrace{div_{\bar{g}} (k - tr_{\bar{g}} k \cdot \bar{g})}_{\bar{\nabla}^i k_{ij} - \bar{\nabla}_j (tr k)} = 8\pi T(\hat{n}, \cdot) & \longrightarrow \text{Momentum constraint} \end{cases}
$$

$(n+1)$ equations!

So: <u>Definition</u>: An initial data set for the vacuum Einstein eqns is a triplet $(\Sigma, \bar{g}, k)$ satisfying ① with $T = 0$.

<u>Remark</u>:

The system ① is undetermined elliptic: For instance, if $\hat{g}$ is any Riemannian metric on Σ and we want to find a "time-symmetric" (i.e. $k = 0$) initial data set, of the form

$$(\bar{g}, k) = (\phi^4 \cdot \hat{g}, 0) \quad (\text{for } n = 3)$$

then ① is equivalent to the following elliptic equation for $\hat{\phi}$:

$$\Delta_{\hat{g}} \phi = \frac{1}{8} \hat{R} \phi$$

Def: Let $(\Sigma, \bar{g}, k)$ be a vacuum initial data set.

Σ: n-manifold, $\bar{g}$: Riem. metric, k: (0,2)-form, symmetric

A development (M, g) of $(\Sigma, \bar{g}, k)$ is a globally hyperbolic spacetime (M,g) solving $Ric = 0$, together with an embedding $i: \Sigma \longrightarrow M$ such that:

- $i(\Sigma)$ is a Cauchy hypersurface of (M,g)
- The induced metric (i.e. $i^* g = \bar{g}$)
- With respect to the fut. directed normal $\hat{n}$ of $i(\Sigma)$, the second fundamental form is k.

Theorem: (Choquet-Bruhat, 1952)

$\forall$ smooth initial data set $(\mathbb{R}^n, \bar{g}, k)$ with $\|\bar{g} - \bar{g}_E\|_{H^{2n+2}} + \|k\|_{H^{2n+1}} < +\infty$,

$\exists$ development $((-\delta, \delta) \times \mathbb{R}^n, g)$ which is geometrically unique.

Sketch of proof:

We want to end up solving $\widetilde{Ric} = 0 \Leftrightarrow \Box_g g_{\mu\nu} = Q_{\mu\nu}(g, \partial g)$.

So we need to choose initial data $g_{\mu\nu}|_{t=0}$, $\partial_t g_{\mu\nu}|_{t=0}$ which are compatible with $(\bar{g}, k)$ and moreover $\Gamma_\mu|_{t=0} = 0$, $\partial_t \Gamma_\mu|_{t=0} = 0$ (so that $\Gamma \equiv 0$).

So at $t=0$: Set $g_{00} = -1$, $g_{0i} = 0$ (so that $\hat{n} = \partial_t$).

Then $g|_{t=0} = \left[\begin{array}{c|ccc} -1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \bar{g} & \\ 0 & & & \end{array}\right]$, $\partial_t g|_{t=0} = \left[\begin{array}{c|ccc} * & * & \cdots & * \\ \hline * & & & \\ \vdots & & 2k & \\ * & & & \end{array}\right]$

(So $\partial_t g_{ij}|_{t=0} = 2k_{ij}$. This is because

$$\partial_t \langle \partial_i, \partial_j \rangle = \langle \nabla_{\partial_t}\partial_i, \partial_j\rangle + \langle \partial_i, \nabla_{\partial_t}\partial_j\rangle = \langle \nabla_{\partial_i}\partial_t, \partial_j\rangle + \langle \partial_i, \nabla_{\partial_j}\partial_t\rangle$$

$$= \langle \nabla_{\partial_i}\hat{n}, \partial_j\rangle + \langle \partial_i, \nabla_{\partial_j}\hat{n}\rangle = k_{ij} + k_{ji}.\ )$$

So $(\bar{g}, k)$ and our choice $\partial_t = \hat{n}$

completely determine $g_{\alpha\beta}|_{t=0}$ $(\alpha,\beta = 0,\dots,n)$ and $\partial_t g_{ij}|_{t=0}$ $(i,j=1,\dots,n)$

We still have the freedom to choose $\partial_t g_{0\alpha}$, $\alpha = 0,\dots,n$.

The requirement that $\Gamma_\gamma|_{t=0} = 0 \iff g^{\alpha\beta}(2\partial_\alpha g_{\gamma\beta} - \partial_\gamma g_{\alpha\beta})|_{t=0} = 0$ for $\gamma = 0,\dots,n$ fixes all the components $\partial_t g_{0\alpha}|_{t=0}$.

There is no more freedom left! How do we ensure that $\partial_t \Gamma_\mu|_{t=0} = 0$?

This is magically implied by the constraint equations!!

Since we are solving $\widetilde{Ric}_{\mu\nu} = 0 \iff Ric_{\mu\nu} = \frac{1}{2}(\partial_\mu \Gamma_\nu + \partial_\nu \Gamma_\mu)$

The constraint equations

$$\left.\begin{aligned} Ric_{00} - \tfrac{1}{2} R \cdot g_{00}|_{t=0} &= 0 \\ Ric_{0i} - \tfrac{1}{2}\underbrace{\cancel{R g_{0i}}}_{0}|_{t=0} &= 0 \end{aligned}\right\} \quad \begin{aligned} \partial_0\Gamma_0 - \tfrac{1}{2}\partial^\mu \Gamma_\mu|_{t=0} &= 0 \\ \partial_0\Gamma_i + \partial_i \Gamma_0|_{t=0} &= 0 \end{aligned}$$

Since $\Gamma_\mu|_{t=0} = 0 \Rightarrow \partial_i \Gamma_\mu|_{t=0} = 0$ for $i = 1,\dots,n$

So the above relations yield

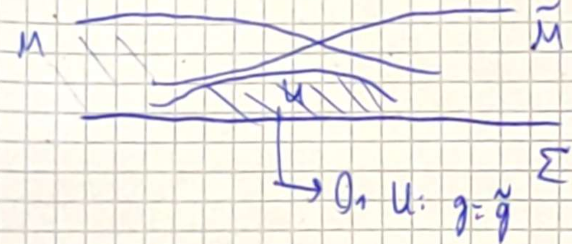
$$\left.\begin{aligned} \partial_0\Gamma_0 + \tfrac{1}{2}\partial_0\Gamma_0|_{t=0} &= 0 \\ \partial_0\Gamma_i|_{t=0} &= 0 \end{aligned}\right\} \partial_0\Gamma_\alpha|_{t=0} = 0$$

for $\alpha = 0,\dots,n$

So from the local well-posedness theory for non-linear wave eqns:
We get existence.

Uniqueness: Suppose that (M, g) and $(\tilde{M}, \tilde{g})$ are two developments.

In wave coordinates ($\Box_g x^\mu = 0$, with $x^0|_\Sigma = 0$ and $x^i|_\Sigma$ fixed so that the embeddings i, $\tilde{i}$ agree, and $\partial_0 x^a|_S$ fixed so that $\partial_0|_S = \hat{n}$): We have two solutions of $\Box_g g_{\mu\nu} = Q_{\mu\nu}(g; \partial g)$ with the same initial data.



So we can "glue" the developments. $\square$

If the initial data are large, or $\Sigma \neq \mathbb{R}^n$:

Restrict in small balls, construct the solution in the corresponding domains of development, glue on the overlaps.

Theorem (Choquet-Bruhat, Geroch '69)
Given a smooth initial data set (Σ, g, k) for the vacuum equations, there exists a unique maximal globally hyperbolic development (M, g)

Maximal: If $(\tilde{M}, \tilde{g})$ is any other developent, then there exists an isometric embedding $F: \tilde{M} \longrightarrow M$ such that $i = F \circ \tilde{i}$

Philosophically: General relativity is a deterministic theory: Initial data determine uniquely the future and the past

Where does the spacetime end?

Is (M, g) geodesically complete if $(\Sigma, \bar{g}, k)$ is?

Answer: No! Example: Schwarzschild ($r=0$ is at finite affine distance)

Next time: Some criteria for incompleteness